

On solvable spherical subgroups of semisimple algebraic groups

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This report contains a new approach to classification of solvable spherical subgroups of semisimple algebraic groups. This approach is completely different from the approach by D. Luna [1] and provides an explicit classification.

Let G be a connected semisimple complex algebraic group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by U the maximal unipotent subgroup of G contained in B . The Lie algebras of G , B , U , ... are denoted by \mathfrak{g} , \mathfrak{b} , \mathfrak{u} , ..., respectively. Let $\Delta \subset \mathfrak{X}(T)$ be the root system of G with respect to T (where $\mathfrak{X}(T)$ denotes the character lattice of T). The subsets of positive roots and simple roots with respect to B are denoted by Δ_+ and Π , respectively. For any root $\alpha \in \Delta_+$ consider its expression of the form $\alpha = \sum_{\gamma \in \Pi} k_\gamma \gamma$. We put

$\text{Supp } \alpha = \{\gamma \in \Pi \mid k_\gamma > 0\}$. The root subspace of the Lie algebra \mathfrak{g} corresponding to a root α is denoted by \mathfrak{g}_α . The symbol $\langle A \rangle$ will denote the linear span of a subset $A \subset \mathfrak{X}(T)$ in $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $H \subset B$ be a connected solvable algebraic subgroup of G and $N \subset U$ its unipotent radical. We say that the group H is *standardly embedded* into B (with respect to T) if the subgroup $S = H \cap T$ is a maximal torus of H . Obviously, in this case we have $H = S \ltimes N$.

Suppose that $H \subset G$ is a connected solvable subgroup standardly embedded into B . Then we can consider the natural restriction map $\tau : \mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$. Put $\Phi = \tau(\Delta_+) \subset \mathfrak{X}(S)$. Then we have $\mathfrak{u} = \bigoplus_{\lambda \in \Phi} \mathfrak{u}_\lambda$, where $\mathfrak{u}_\lambda \subset \mathfrak{u}$ is the weight subspace of weight λ with respect to S . Similarly, we have $\mathfrak{n} = \bigoplus_{\lambda \in \Phi} \mathfrak{n}_\lambda$, where $\mathfrak{n}_\lambda = \mathfrak{u}_\lambda \cap \mathfrak{n} \subset \mathfrak{u}_\lambda$.

Denote by c_λ the codimension of \mathfrak{n}_λ in \mathfrak{u}_λ .

Recall that a subgroup $H \subset G$ is called *spherical* if the group B has an open orbit in G/H . The following theorem provides a convenient criterion of sphericity for connected solvable subgroups of G .

Theorem 1. *Suppose $H \subset G$ is a connected solvable subgroup standardly embedded into B . Then the following conditions are equivalent:*

- (1) *H is spherical in G ;*
- (2) *$c_\lambda \leq 1$ for every $\lambda \in \Phi$, and the weights λ with $c_\lambda = 1$ are linearly independent in $\mathfrak{X}(S)$.*

Now we suppose that H is spherical.

Definition 1. A root $\alpha \in \Delta_+$ is called *marked* if $\mathfrak{g}_\alpha \not\subset \mathfrak{n}$.

An important role of marked roots in studying solvable spherical subgroups is clear from the theorem below.

Theorem 2. *Up to conjugation by elements of T , the subgroup H is uniquely determined by its maximal torus $S \subset T$ and the set $\Psi \subset \Delta_+$ of marked roots.*

Remark 1. The subgroup H is explicitly recovered from S and Ψ .

Marked roots have the following property: if $\alpha \in \Psi$ and $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta_+$, then exactly one of two roots β, γ is marked. Taking this property into account, we say that a marked root β is *subordinate* to a marked root α , if $\alpha = \beta + \gamma$ for some root $\gamma \in \Delta_+$. Given a marked root α , we denote by $C(\alpha)$ the set consisting of α and all marked roots subordinate to α . Further, we say that a marked root α is *maximal* if it is not subordinate to any other marked root. Let M denote the set of all maximal marked roots.

Proposition 1. *For any marked root α there exists a unique simple root $\pi(\alpha) \in \text{Supp } \alpha$ with the following property: if $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta_+$, then the root β is marked iff $\pi(\alpha) \notin \text{Supp } \beta$ (and so the root γ is marked iff $\pi(\alpha) \notin \text{Supp } \gamma$).*

Definition 2. If α is a marked root, then the simple root $\pi(\alpha)$ appearing in Proposition 1 is called the *simple root associated with the marked root α* .

From Proposition 1 we see that for any marked root α the set $C(\alpha)$ is uniquely determined by the simple root $\pi(\alpha)$. Therefore, the whole set Ψ is uniquely determined by the set M and the map $\pi: M \rightarrow \Pi$.

Theorem 3 (Classification of marked roots). *All possibilities for a marked root α and the simple root $\pi(\alpha)$ are presented in Table 1.*

Notations used in Table 1. The symbol $\Delta(\alpha)$ denotes the root system generated by $\text{Supp } \alpha$, i. e. $\Delta(\alpha) = \langle \text{Supp } \alpha \rangle \cap \Delta$. We suppose that $\text{Supp } \alpha = \{\alpha_1, \dots, \alpha_n\}$. The numeration of simple roots of simple Lie algebras is the same as in [2].

Table 1: Marked roots

	type of $\Delta(\alpha)$	α	$\pi(\alpha)$
1	any of rank n	$\alpha_1 + \alpha_2 + \dots + \alpha_n$	$\alpha_1, \alpha_2, \dots, \alpha_n$
2	B_n	$\alpha_1 + \alpha_2 + \dots + \alpha_{n-1} + 2\alpha_n$	$\alpha_1, \alpha_2, \dots, \alpha_{n-1}$
3	C_n	$2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$	α_n
4	F_4	$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	α_3, α_4
5	G_2	$2\alpha_1 + \alpha_2$	α_2
6	G_2	$3\alpha_1 + \alpha_2$	α_2

For any root $\alpha \in \Delta_+$ consider the (connected) Dynkin diagram $D(\alpha)$ of the set $\text{Supp } \alpha$. We say that a root $\delta \in \text{Supp } \alpha$ is *terminal with respect to $\text{Supp } \alpha$* if the node of $D(\alpha)$ corresponding to δ is connected by an edge (possibly, multiple) with exactly one other node of $D(\alpha)$.

Now we introduce some conditions on a pair of marked roots (α, β) . These

conditions will be used later.

(D0) $\text{Supp } \alpha \cap \text{Supp } \beta = \emptyset$

(D1) $\text{Supp } \alpha \cap \text{Supp } \beta = \{\delta\}$, δ is terminal with respect to both $\text{Supp } \alpha$ and $\text{Supp } \beta$, $\pi(\alpha) \neq \delta \neq \pi(\beta)$

(D2) the Dynkin diagram of the set $\text{Supp } \alpha \cup \text{Supp } \beta$ has the form shown on Figure 1 (for some $p, q, r \geq 1$), $\alpha = \alpha_1 + \dots + \alpha_p + \gamma_0 + \gamma_1 + \dots + \gamma_r$, $\beta = \beta_1 + \dots + \beta_q + \gamma_0 + \gamma_1 + \dots + \gamma_r$, $\pi(\alpha) \notin \text{Supp } \alpha \cap \text{Supp } \beta$, $\pi(\beta) \notin \text{Supp } \alpha \cap \text{Supp } \beta$

(E1) $\text{Supp } \alpha \cap \text{Supp } \beta = \{\delta\}$, δ is terminal with respect to both $\text{Supp } \alpha$ and $\text{Supp } \beta$, $\delta = \pi(\alpha) = \pi(\beta)$, $\alpha - \delta \in \Delta_+$, $\beta - \delta \in \Delta_+$

(E2) the Dynkin diagram of the set $\text{Supp } \alpha \cup \text{Supp } \beta$ has the form shown on Figure 1 (for some $p, q, r \geq 1$), $\alpha = \alpha_1 + \dots + \alpha_p + \gamma_0 + \gamma_1 + \dots + \gamma_r$, $\beta = \beta_1 + \dots + \beta_q + \gamma_0 + \gamma_1 + \dots + \gamma_r$, $\pi(\alpha) = \pi(\beta) \in \text{Supp } \alpha \cap \text{Supp } \beta$

Next, we need to introduce an equivalence relation on M . For any two roots $\alpha, \beta \in M$ we write $\alpha \sim \beta$ iff $\tau(\alpha) = \tau(\beta)$. Having introduced this equivalence relation, to each connected solvable spherical subgroup H standardly embedded into B we assign the set of combinatorial data (S, M, π, \sim) .

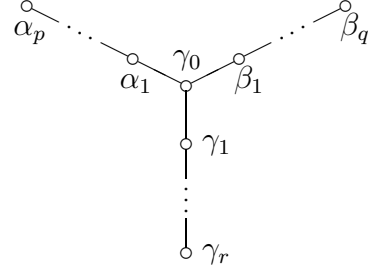


Figure 1.

Theorem 4. *The above assignment is a one-to-one correspondence between the following two sets:*

- (1) *the set of all connected solvable spherical subgroups standardly embedded into B , up to conjugation by elements of T ;*
- (2) *the set of all sets (S, M, π, \sim) , where $S \subset T$ is a torus, M is a subset of Δ_+ , $\pi: M \rightarrow \Pi$ is a map, \sim is an equivalence relation on M , and the set (S, M, π, \sim) satisfies the following conditions:*

- (A) $\pi(\alpha) \in \text{Supp } \alpha$ for any $\alpha \in M$, and the pair $(\alpha, \pi(\alpha))$ is contained in Table 1;
- (D) if $\alpha, \beta \in M$ and $\alpha \not\sim \beta$, then one of conditions (D0), (D1), (D2) holds;
- (E) if $\alpha, \beta \in M$ and $\alpha \sim \beta$, then one of conditions (D0), (D1), (E1), (D2), (E2) holds;

(C) *for any $\alpha \in M$ the condition $\text{Supp } \alpha \not\subset \bigcup_{\beta \in M \setminus \{\alpha\}} \text{Supp } \beta$ holds;*

(T) $\text{Ker } \tau|_R = \langle \alpha - \beta \mid \alpha, \beta \in M, \alpha \sim \beta \rangle$, where $R = \langle \bigcup_{\gamma \in M} \text{Supp } \gamma \rangle$.

Remark 2. The unipotent radical $N \subset U$ of a connected solvable spherical subgroup H standardly embedded into B is uniquely (up to conjugation by elements of T) determined by the set (M, π, \sim) satisfying conditions (A), (D), (E), (C).

To complete the classification of connected solvable spherical subgroups of G up to conjugation, it remains to determine all sets of combinatorial data that correspond to one conjugacy class of such subgroups. Consider again a connected solvable spherical subgroup $H \subset G$ standardly embedded into B . We say that a marked root α is *regular* if the projection of \mathfrak{n} to the root space \mathfrak{g}_α is zero. Choose any regular marked simple root $\alpha \in \Delta_+$ and fix an element $n_\alpha \in N_G(T)$ such that

its image in the Weyl group W is the simple reflection r_α corresponding to α (here $N_G(T)$ is the normalizer of T in G). Obviously, the group $n_\alpha H n_\alpha^{-1}$ is also standardly embedded into B . Its set of combinatorial data is $(n_\alpha S n_\alpha^{-1}, M', \pi', \sim')$ for some M' , π' , and \sim' . In order to determine M' , π' , and \sim' , we consider two cases:

(1) if $\alpha \in \text{Supp } \delta$ for some $\delta \in r_\alpha(M \setminus \{\alpha\})$, then $M' = r_\alpha(M \setminus \{\alpha\})$, $\pi'(\beta) = \pi(r_\alpha(\beta))$ for any $\beta \in M'$, $\beta \sim' \gamma$ iff $r_\alpha(\beta) \sim r_\alpha(\gamma)$ for any $\beta, \gamma \in M'$;

(2) if $\alpha \notin \text{Supp } \delta$ for any $\delta \in r_\alpha(M \setminus \{\alpha\})$, then $M' = r_\alpha(M \setminus \{\alpha\}) \cup \{\alpha\}$, $\pi'(r_\alpha(\beta)) = \pi(\beta)$ for any $\beta \in M \setminus \{\alpha\}$, $\pi'(\alpha) = \alpha$, $r_\alpha(\beta) \sim' r_\alpha(\gamma)$ iff $\beta \sim \gamma$ for any $\beta, \gamma \in M \setminus \{\alpha\}$, $\alpha \not\sim' \beta$ for any $\beta \in M' \setminus \{\alpha\}$.

Transformations of the form $H \mapsto n_\alpha H n_\alpha^{-1}$ described above are called *elementary transformations*.

Theorem 5. *Suppose $H_1, H_2 \subset G$ are two connected solvable spherical subgroups standardly embedded into B , and $H_2 = g H_1 g^{-1}$ for some $g \in G$. Then:*

- (1) $H_2 = \sigma H_1 \sigma^{-1}$ for some $\sigma \in N_G(T)$;
- (2) H_2 can be obtained from H_1 by applying a finite sequence of elementary transformations.

Thus, Theorems 4 and 5 give a complete classification of connected solvable spherical subgroups of G .

References

- [1] D. Luna, *Sous-groupes sphériques résolubles*, Prépublication de l'Institut Fourier no. 241, 1993.
- [2] A.L. Onishchik, E.B. Vinberg, *Lie Groups and Algebraic Groups*. Springer-Verlag (1990).